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THE PACKING CHROMATIC NUMBER OF HARARY GRAPHS

B. Chaluvaraju[*1], M. Kumara[2]
[*1, 2]Department of Mathematics, Bangalore University, Jnana Bharathi Campus, Bengaluru - 560 056, INDIA

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ABSTRACT

A packing k-coloring of a graph $G = (V, E)$ is a partition of V(G) in to sets $V_1, V_2, \ldots, V_k$ such that for each $1 \leq i \leq k$ the distance between any two distinct $u, v \in V_i$ is atleast $i+1$. The packing chromatic number $\chi_\rho(G)$ is the minimum k such that G has a packing k-coloring. In this paper, we compute the packing chromatic number of the Harary graphs.

KEYWORDS: Coloring; Packing chromatic number; Harary graph.

1. INTRODUCTION

Throughout this paper, the graphs chosen are finite, undirected, simple and connected.

A graph $G = (V, E)$ with $p = |V|$ and $q = |E|$ denote the number of vertices and edges, respectively. For graph-theoretical terminology and notation not defined here we follow [6].

A coloring of G is defined to be an assignment of colors to its vertices so that no pair of adjacent vertices shares identical colors. In other words, a coloring of G is a mapping $f : V(G) \to \{1, 2, \cdots, p\}$ such that, for any $(i, j) \in E(G), f(i) \neq f(j)$. A coloring induces natural a partition of $V(G)$ such that the elements of each set in the partition are pair-wise nonadjacent; these sets are precisely the subsets of vertices being assigned the same color. If there exists a coloring of G that uses no more than k colors, G admits a k-coloring (G is k-colorable). The minimal k for which G admits a k-coloring is called the chromatic number and is denoted by $\chi_\rho(G)$. For complete review on theory of coloring and its related concepts, we refer to [8] and [10].

In graph terms, we ask for a partition of the vertex set of a graph G into disjoint classes $X_1, X_2, \ldots, X_k$ (representing frequency usage) according to the following constraints. Each color class X_i should be an i - packing, that is, a set of vertices with the property that any distinct pair $u, v \in X_i$ satisfies $dist(u, v) > i$. Here $dist(u, v)$ denotes the usual shortest path distance between u and v. Such a partition is called a packing k-coloring, even though it is allowed that some sets X_i may be empty. The smallest integer k for which there exists a packing k-coloring of G is called the packing chromatic number of G and it is denoted by $\chi_\rho G)$. This concept was introduced by Goddard et al.,[5] under the name broadcast chromatic number.

The term packing chromatic number was later (even if the corresponding paper was published earlier) proposed by Bresar *et al*., [2]. For complete review of packing chromatic number we refer to [4], [11] and [12].

In 1962, Harary introduced the Harary graph, which has maximum connectivity; therefore, it plays an important role in designing networks. This is a m-connected graph on p vertices of degree at least m with $\frac{mp}{2}$ edges.

The Harary graph $H_{m,p}$ is constructed as follows:

(i) m even. Let $m = 2r$. Then $H_{2r,p}$ is constructed as follows. It has vertices $0, 1, 2, \ldots, p-1$ and two vertices i and j are joined if $i - r \leq j \leq i + r$.

(ii) m odd, p even. Let $m = 2r + 1$. Then $H_{2r+1,p}$ is constructed by first drawing $H_{2r,p}$ and then adding edges joining vertex i to vertex $i+\frac{p}{2}$ for $1 \le i \le \frac{p}{2}$.

(iii) m odd, p odd. Let $m = 2r + 1$. Then $H_{2r+1,p}$, p is constructed by first drawing $H_{2r,p}$ and then adding edges joining vertex 0 to vertices $\frac{(p-1)}{2}$ and $\frac{(p+1)}{2}$ and vertex i to vertex $i + \frac{(p+1)}{2}$ for $1 \le i \le \frac{(p-1)}{2}$.

A communication network constructed as a Harary graph $H_{m,p}$ tells us that we can remove up to p vertices before the network becomes partitioned or in other words the network is split into components. This means that if we are considering networks that are designed to disseminate data to every node, the Harary graphs will give us the means to make them just as robust as we want them to be, yet with a minimal number of links. For more details, we refer to [7] and [9].

For a given positive integer p, let $s_1, s_2, \ldots, s_t$ be a sequence of integers where

$$0 < s_1 < s_2 < \ldots. < s_t < \frac{p+1}{2}.$$

Then the circulant graph $C_p(s_1, s_2, \ldots, s_t)$ is the graph on p vertices labeled as $v_1, v_2, \ldots, v_p$ with vertex v_i adjacent to each vertex $v_{i\pm sj\ (mod\ p)}$ and the values s_t are called jump sizes. Circulant graph belongs to the family of Cayley graphs, which are used as models for interconnection networks in telecommunication, VLSI designs, parallel and distributed computing. For more details, we refer to [1] and [3].

2. RESULTS

Theorem 2.1. Let $G = H_{m;p}$ be a Harary graph with $m = 2r$ and $p = 4r$; $r \ge 1$. Then

$$\chi_\rho(H_{m,p}) = \begin{cases} p-1, if p = 4r; r \le 2 \\ p-2 if p = 4r; r \ge 3 \end{cases}$$

Proof. Let $G = H_{m,p}$ be a Harary graph with p vertices, where $m = 2r$ and $p = 4r$, $r \ge 1$. Therefore, the vertex set of $H_{m,p}$ is given by $V(H_{m,p}) = \{v_0, v_1, v_2, \ldots, v_{p-1}\}$, we know that the graph $H_{m,p}$ contains p vertices and the degree of each vertex is m. By the definition of harary graph we know that any two vertices v_i and v_j are joined if $i-p \le j \le i + p$. Then the following two cases.

Case 1. $p = 4r$, $r \le 2$.
Let us now see the procedure to packing color the graph $H_{m,p}$ as follows. Assign colors of vertices $v_1, v_2, \ldots, v_p$ with the distinct color classes $X_1, X_2, \ldots, X_k$. In which the sequence 12,13,... starting from v_1 to v_p. Thus, there is a proper packing coloring and therefore $\chi_\rho(H_{m,p}) = p - 1$.

Case 2. $p = 4r$, $r \ge 3$.
Consider the graph with vertices $p = 4r$, $r \ge 3$, assign the color to the vertices of graph $H_{m,p}$. In which the color from vertex v_1 assign color 1 and vertex v_2 assign color 2 likewise in a sequence with distinct pair and distance greater than 1. Therefore, $\chi_\rho(H_{m,p}) = p - 2$.

The following result are obtained by using Theorem 2.1. The jump size of Circulant graph is one, known as cycle C_p with $p \ge 4$ vertices. That is, $C_p(1) \approx C_p$; $p \ge 4$.

Corollary 2.1. For $m = 2$ and $p \ge 3$ vertices,

$$\chi_\rho(H_{m,p}) = \chi_\rho(C_p) = \chi_\rho(C_p(1)).$$

Theorem 2.2. Let $H_{m,p}$ be a Harary graph with $m = 2r + 1$, $r \geq 3$ and p even. Then

$$\chi_\rho(H_{m,p}) = p.$$

Proof. Let $H_{m,p}$ be the harary graph with the parities m and p and take that m is odd and p is even. Let us take $m = 2r + 1$ and $p = 2r + 2$. By the definition of harary graphs for the case m-odd and p-even is constructed as follows. First $H_{2r,2r+2}$ is constructed. Then edges are joined from vertex *i to vertex i* $+\left(\frac{2r+2}{2}\right)$ for $1 \leq i \leq \left(\frac{2r+2}{2}\right)$.

Let us now assign colors to packing color the graph. We can observe that the harary graph $H_{2r+1,2r+2}$ to be a complete graph with $2r + 2$ vertices. Since every pair of vertices is adjacent to each other. In a complete graph, since all the vertices are adjacent, each vertex receives different colors. Thus, for any vertex v_i its neighborhood vertices are assigned with distinct colors. Therefore, for any path on four vertices is not bicolored thus the packing chromatic number is equal to the chromatic number. Since the harary graph $H_{2r+1,2r+2}$ is a complete graph, the above discussion also holds for the harary graph $H_{2r+1,2r+2}$. Thus the packing chromatic number of the harary graph $H_{2r+1,2r+2}$ is equal to the number of vertices. Here p denotes the number of vertices. Therefore $\chi_\rho(H_{m,p}) = p$. Hence the result follows.

The following result are obtained by using Theorem 2.2. The jump size of circulant graph is $1,2,\ldots,\left\lfloor\frac{p}{2}\right\rfloor$, known as complete graph K_p with $p \geq 3$ vertices, that is, $C_p(1,2,\ldots,\left\lfloor\frac{p}{2}\right\rfloor) \cong K_p$.

Corollary 2.2. For $m = p - 1$ and $p \geq 3$,

$$\chi_\rho(H_{m,p}) = \chi_\rho(K_p) = \chi_\rho(Cp(1,2,\ldots,\left\lfloor\frac{p}{2}\right\rfloor)).$$

Theorem 2.3. Let $H_{m,p}$ be a Harary graph with $m = 2r + 1$, $r \geq 2$ and $p = 4r + 1$ and p odd. Then

$$\chi_\rho(H_{m,p}) = p - 1.$$

Proof. Let $H_{m,p}$ be the harary graph with the parities m and p and take that m and p is odd. Let us take $m = 2r + 1$ and $p = 4r + 1$. By the definition of harary graphs for the case m and p odd is constructed as follows. First $H_{2r+1,4r+1}$ is constructed. Then edges are joined from vertex i to vertex $i + \left(\frac{4r+1}{2}\right)$ for $1 \leq i \leq \left(\frac{4r+1}{2}\right)$. Let us now assign colors to packing color the graph. Since the harary graph $H_{2r+1,4r+1}$ has been assigned distinct colors from vertex v_1 to v_p. Here one can observe that the vertex set $V(G)$ can be partitioned into disjoint color classes $X_1, X_2,\ldots,X_k$, where vertices in X_i have pairwise distance greater than i. Therefore $\chi_\rho(H_{m,p}) = p - 1$ where $m = 2r + 1$ and $p = 4r + 1$; $r \geq 2$.

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